DIFFERENTIAL MANIFOLDS HW 6

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1. Exercise 3.9

Suppose $f : M \to N$ is a proper injective immersion, where M, N are Hausdorff second countable. Then, given $p \in M$, there is a chart on N reducing to the canonical injection in some open neighborhood on p. In order to prove that this is a homeomorphism onto $f(M)$, it suffices to prove that every basic neighborhood contains a subset such that f is a homeomorphism when restricted to this subset.

Since M, N are locally homeomorphic to Euclidean n-space, they are locally compact, and hence paracompact. Then, atlases are locally finite so we can shrink our neighborhood such that it is contained within a single chart domain for both M, N, we see that the image of an open set $f(U)$ is identically $a^{-1}(U, 0)$, which is open in the induced topology $f(M)$. Hence, restricting to a sufficiently small neighborhood, f is an embedding.

By local compactness, we can find a compact $K \subset N$ such that $f(p) \in K^o$. Then, $f^{-1}(K)\backslash U$ is the intersection of a compact set and a closed set, hence compact. Extract a finite subcover $\{U_1, \dots, U_n\}$ of some open cover of $f^{-1}(K)\backslash U$. Then, the images $f(U_i) \subset N$ do not intersect $f(p)$ by injectivity, and hence we can find an open set $V \ni f(p)$ disjoint from each $f(U_i)$ such that $V \subset K^o$.

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If we can prove that $f|_{f^{-1}(V)}$ is a homeomorphism, then we are done. To do this, it suffices to prove that $f^{-1}(V) \subset U$. But, V was chosen disjoint from each $f(U_i)$, which in turn were chosen disjoint from U, so in particular $f^{-1}(V) \subset U\backslash(f^{-1}(K)\backslash U) = U$. Hence, $f^{-1}(V) \subset U$ so that $f|_{f^{-1}(V)}$ is a homeomorphism. Extending this argument to every point, f is a topological embedding at every point and hence an embedding.

2. Problem 2

Define $L : G \times X \to X$ by sending $(g, x) \mapsto g \cdot x$, $L_g : X \to X$, $L_g(x) = g \cdot x$, and $L^x : G \to X$, $L^x(g) = g \cdot x$ (where $\cdot$ denotes our action).

Using this, first note that $Z_G(g) \in T_gG$, as we can write $Z_G(g) = T_0e^{tZ}g\Big(\frac{\partial}{\partial u}|_t\Big)$. Then, for $\delta x \in T_xX$:

$$T_{(g,x)}L(Z_G(g), \delta x) = T_gL^x(Z_G(g)) + T_xL_g(\delta x)$$

Then, noting that $e^{tZ}g|_{t=0} = g$, we see:

$$\begin{aligned} T_gL^x(Z_G(g)) &= T_gL^xT_0e^{tZ}\Big(\frac{\partial}{\partial u}\Big|_t\Big) \\ &= T_0(L^x \circ (e^{tZ}g)\Big(\frac{\partial}{\partial u}\Big|_t\Big) \\ &= \frac{d}{dt}\Big|_{t=0} e^{tZ}g \cdot x = Z_X(g \cdot x) \end{aligned} \tag{2.1}$$

Then, putting this together:

$$T_{(g,x)}L(Z_G(g), \delta x) = Z_X(g \cdot x) + T_xL_g(\delta x)$$

As asserted.

3. Problem 3 (Remark after Proposition 5.111)

We first recall the trivial fact that on a set with the discrete topology, a subset is compact if and only if it is finite. Hence we want to prove the equivalence of the definition given in 1.106 with Proposition 5.111.

We want to show 1.106 $\implies$ 5.111. Assuming 1.106 holds, suppose K is compact. Then, for every $x \in K$, there exists an open neighborhood $U_x \ni x$. Then, taking the union over all $x \in K$, this constitutes and open and by compactness we can find a finite subcover $\{U_{x_1}, \ldots, U_{x_n}\}$.

Denote by $G_{x_i x_j} := \{g \in G : \ gU_{x_i} \cap U_{x_j} \neq \varnothing\}$. By assumption, this set is always finite. It is also clear that G_K is contained in $\bigcup_{i,j} G_{x_i x_j}$ and is hence also always finite $\implies$ G_K is compact. Since K was arbitrary, the result follows.

Conversely, argue by contraposition. Suppose we can find $x, y \in M$ such that $\{g \in G : \ gU_x \cap U_y \neq \varnothing\}$ is infinite for all open neighborhoods U_x, U_y. By local compactness, we can find a compact K containing $U_x \cup U_y$. Then, it is clear that $G_K \supset \{g \in G : \ gU_x \cap U_y \neq \varnothing\}$ and is hence also not finite.

4. Problem 4

We wish to solve the system of ODE's given by the flow of the vector field $X := -p^2 q \frac{\partial}{\partial p} + p(1+q^2)\frac{\partial}{\partial q}$. Consider the closed 2-form $\sigma := \mathrm{d}p \wedge \mathrm{d}q$. Then, we see:

$$\begin{aligned}
\mathrm{d}i_x\sigma &= \mathrm{d}(\mathrm{d}p(X) \wedge \mathrm{d}q - \mathrm{d}q(X) \wedge \mathrm{d}p) \\
&= \mathrm{d}(-p^2 q \mathrm{d}q - p(1+q^2)\mathrm{d}p) \qquad (4.1) \\
&= -2pq\mathrm{d}p \wedge \mathrm{d}q - 2pq\mathrm{d}q \wedge \mathrm{d}p = 0
\end{aligned}$$

Hence, σ preserves our vector field and it is a simple computation to see that X is the symplectic gradient of $H(p,q) = \frac{1}{2}(p^2(1+q^2) + c$, c is any constant. Thus $L_X H = 0$. But $L_X H = \mathrm{d}H(X)$. Hence, given an integral $\gamma(t) = (p(t), q(t))$ of X:

$$\mathrm{d}H_{\gamma(t)}(X(\gamma(t))) = \mathrm{d}H_{\gamma(t)}(\gamma'(t)) = 0$$

Hence, we want to calculate $\mathrm{d}H_{\gamma(t)}(\gamma'(t))$:

$$\mathrm{d}H_{\gamma(t)}(\gamma'(t)) = -p^2 q q' - p(1+q^2)p'$$

Setting this equal to 0, we find that $\frac{qq'}{(1+q^2)} = \frac{p'}{p}$, and integrating:

$$p = A(1+q^2)^{-1/2}$$

Where $A = p_0(1+q_0^2)^{1/2}$. Using this, we can solve first for q by integrating the relation $q'(t) = p(1+q^2) = A(1+q^2)^{1/2}$. We see:

$$q(t) = \sinh(At + B)$$

Where B is some constant. Then, using the fact that $p = A(1+q^2)^{-1/2}$:

$$p(t) = \frac{A}{(1+\sinh^2(At+B))^{1/2}} = A\,\mathrm{sech}(At+B)$$

So our solution is the curve

$$\gamma(t) = \Big(\sinh(At+B), A\,\mathrm{sech}(At+B)\Big)$$

5. Problem 5

Let $X := \{x \in \mathbb{C}^{n\times n} : \ x^2 = x = \overline{x}; \ \mathrm{Tr}(x) = k\}$.

(a). Let

$$x_0 = \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}$$

Where I_K is the $k \times k$ identity. Then, we wish to show that for $G = U(n)$ acting by conjugation, $G \cdot x_0 = X$. To do this, let $x \in X$ arbitrary and choose an orthonormal basis $\{u_1, \ldots, \ u_k\}$ for the range of x in $\mathbb{C}^n$ and complete this to a basis for the whole space.

Then, $x = u_1\overline{u}_1 + \cdots + u_k\overline{u}_k$. Set $g = (u_1 \ u_2 \ \ldots \ u_n)$. Then, g is unitary as our u_i are orthonormal and if e_i denotes the standard basis vectors (for $\mathbb{C}^n$), the $ge_i = u_i$. However, $x_0 = e_1\overline{e}_1 + \cdots + e_k\overline{e}_k$, and hence:

$$\begin{aligned} gx_0\overline{g} &= ge_1\overline{e}_1\overline{g} + \cdots + ge_k\overline{e}_k\overline{g} \\ &= ge_1\overline{ge_1} + \cdots + ge_k\overline{ge_k} \\ &= u_1\overline{u}_1 + \cdots + u_k\overline{u}_k = x \end{aligned} \tag{5.1}$$

Since $x \in X$ was arbitrary, we conclude that $G \cdot x_0 = X$.

(b). We first compute the Lie Algebras of our respective Lie Groups G and G_{x_0}. In the first case, we have that $x\overline{x} = I$, and hence deriving and setting $\delta x = I_n$ gives $x + \overline{x} = 0$, so that $x = -\overline{x}$. Then the Lie Algebra of $G := U(n)$ is the subset of skew-adjoint matrices.

Similarly, we can derive the relation $gx_0\overline{g} = x_0$ for each $g \in G_{x_0}$. Doing this yields

$$\delta g x_0 \overline{g} + g x_0 \overline{\delta g} = 0$$

Setting $\delta g = I_n$, we compute the Lie Algebra as the set of g with $g x_0 g = -x_0$.

Since X is an orbit, it is isomorphic to G/G_{x_0}, and hence we can compute the dimension as $\dim X = \dim G - \dim G_{x_0}$. The dimension of G is simply n^2, and the dimension of G_{x_0} can be found by characterizing the stabilizer of x_0 under the conjugate action. We find that any $g \in G_{x_0}$ is of the form

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

Where A and B are $k \times k$ and $(n-k) \times (n-k)$ unitary matrices, respectively. Then the dimension is easy to calculate as just $k^2 + (n-k)^2$. Subtracting this from $\dim G$, we have:

$$\dim X = n^2 - (n-k)^2 - k^2 = 2k(n-k)$$

Hence $\dim X = 2k(n-k)$. Now we finally wish to compute the infinitesimal action on X. This fairly simple though by using matrix exponentiation. $\gamma(t) = e^{tZ}$ is an arbitrary curve through the identity such that $\gamma'(0) = Z$. Then the infinitesimal action is merely:

$$\frac{d}{dt}\Big|_0 (e^{tZ} \cdot x)(0) = \frac{d}{dt}\Big|_0 (e^{tZ} x e^{-tZ})(0) = Zx - xZ$$

So the infinitesimal action is just taking the commutator.

(c). Note first that $\delta x = x\delta x + \delta x x$. Then we have that $x \delta x x = 2x\delta x x$ so that $x \delta x x = 0$. Remembering this:

$$
\begin{aligned}
-J^2\delta x &= [x,[x,\delta x]] \\
&= x[x,\delta x] - [x,\delta x]x \\
&= x\delta x - x\delta x x - x\delta x x + \delta x x \\
&= x\delta x + \delta x x = \delta x
\end{aligned} \tag{5.2}
$$

Hence, $J^2\delta x = -\delta x$ so that $J^2 = -\mathrm{id}$. Now we can define $\sigma(\delta x, \delta' x) := \mathrm{Tr}(\delta' x J \delta x)$. To show this gives a symplectic structure, we need to prove σ is closed and nondegenerate.

On $U(n)$, we have that $\langle x, Z\rangle = \frac{1}{i}\mathrm{Tr}(xZ)$ is the coadjoint action on its Lie Algebra. Suppose that $x \in X$ is such that $x = gx_o\overline{g}$ for some $g \in G$. Then choose $Z := \delta x x - x\delta x$. Then this is skew-adjoint by construction and we see:

$$
Z(x) = Zx - xZ = \delta x x^2 - x\delta x x - x\delta x + x^2\delta x = \delta x x + x\delta x = \delta x
$$

Z' is defined similarly (just put a $'$ after the δ). A straightforward (but tedious) computation yields that

$$
\langle x, [Z', Z]\rangle = \frac{1}{i}\mathrm{Tr}(x[Z', Z]) = \frac{1}{i}\mathrm{Tr}(\delta' x J \delta x)
$$

So that the above is the KKS 2-form. This immediately tells us that our form is closed and nondegenerate by the KKS classification.